WEIGHTED EHRHART THEORY AND ORBIFOLD COHOMOLOGY

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ABSTRACT. We introduce the notion of a weighted δ-vector of a lattice polytope. Although the definition is motivated by motivic integration, we study weighted δ-vectors from a combinatorial perspective. We present a version of Ehrhart Reciprocity and prove a change of variables formula. We deduce a new geometric interpretation of the coefficients of the Ehrhart δ-vector. More specifically, they are sums of dimensions of orbifold cohomology groups of a toric stack.

1. INTRODUCTION

Let P be a d-dimensional lattice polytope in a lattice N of rank d. For each positive integer m, let $f_P(m)$ be the number of lattice points in mP. Then $f_P(m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of P [9, 10]. The generating series of the Ehrhart polynomial can be written in the form

$$\sum_{m \geq 0} f_P(m)t^m = \delta_P(t)/(1-t)^{d+1},$$

where $\delta_P(t)$ is a polynomial of degree less than or equal to d with non-negative integer coefficients [27]. With a slight abuse of terminology, $\delta_P(t)$ is called the *(Ehrhart) δ-vector* of P. We will write $\delta_P(t) = \delta_d t^d + \delta_{d-1} t^{d-1} + \cdots + \delta_0$.

We present a new geometric interpretation of the coefficients δ_i of the Ehrhart δ-vector. When P is reflexive, Batyrev and Dais [3] showed that δ_i is the $2i$th stringy Betti number of a toric variety X associated to P. Furthermore, in this case, results of Yasuda [31] imply that δ_i is equal to the dimension of the $2i$th orbifold cohomology group of the canonical orbifold associated to X. This interpretation of the Ehrhart δ-polynomial of P was used by Mustaţă and Payne in [23] and Karu in [18]. We generalise this result to any lattice polytope P by showing that δ_i is a sum of dimensions of orbifold cohomology groups of a toric stack (Theorem 4.3).

In order to establish this result, we introduce certain refinements of the Ehrhart polynomial. Fix a lattice point α in P. After translating, we may assume that α is the origin. Denote the union of the facets of P not containing the origin by ∂P_0. Consider the fan $\triangle$ over the faces of ∂P_0, with support $|\triangle|$. Let $\psi : |\triangle| \to \mathbb{R}$ be the piecewise $\mathbb{Q}$-linear function with respect to $\triangle$ satisfying $\psi(v) = 1$ for all v in ∂P_0. We fix a simplicial fan Σ refining $\triangle$ and with the same rays as $\triangle$ (see, for example, [24]). Let $\lambda : |\Sigma| \to \mathbb{R}$ be an arbitrary piecewise $\mathbb{Q}$-linear function with respect to Σ satisfying $\lambda(v) > -1$ for all v in ∂P_0. Consider the function

$$w_\lambda : |\Sigma| \cap N \to \mathbb{Q}$$

$$w_\lambda(v) = \psi(v) - \lceil \psi(v) \rceil + \lambda(v).$$

Note that when $\lambda \equiv 0$, the function w_0 is determined by the pair (P, α). Think of w_λ as assigning a weight to every lattice point in $|\Sigma|$. For every rational number k and for every non-negative integer m, denote by $f_k^\lambda(m)$ the number of lattice points of weight k in mP. Note that the Ehrhart polynomial of P can be recovered as $f_P(m) = \sum_{k \in \mathbb{Q}} f_k^\lambda(m)$. For every rational number k, consider the power series

$$\delta_k^\lambda(t) := (1-t)^{d+1} \sum_{m \geq 0} f_k^\lambda(m) t^m.$$

If $\lambda(v) \geq 0$ for all v in $|\Sigma| \cap N$, then $\delta_k^\lambda(t)$ is a polynomial in t with integer coefficients (Corollary 2.9). The δ-vector of P decomposes as $\delta_P(t) = \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t)$. We define the *weighted δ-vector* of P to be

$$\delta^\lambda(t) := \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t) t^k.$$

In Section 2, we verify that $\delta^\lambda(t)$ lies in $\mathbb{Z}[[t^{1/N}]]$, for some positive integer N. We discuss the geometric interpretation of the weighted δ-vector in terms of motivic integration in [30].

We emphasise that the weighted δ-vector of P is interesting from a purely combinatorial perspective. We present a change of variables formula in Section 2, comparing weighted δ-vectors on different polytopes (Proposition 2.13). The weighted δ-vector has the following symmetry property (Corollary 2.12).

Proposition. *If the origin lies in the interior of P, then the weighted δ-vector $\delta^\lambda(t)$ satisfies*

$$\delta^\lambda(t) = t^d \delta^\lambda(t^{-1}).$$

The most important example is when $\lambda \equiv 0$, since the Ehrhart δ-vector $\delta_P(t)$ can be easily recovered from the weighted δ-vector $\delta^0(t)$ (see (14)). More specifically, $\delta^0(t)$ is a polynomial of degree d with rational powers and non-negative integer coefficients (see (15)) and the coefficient of t^i in $\delta_P(t)$ is the sum of the coefficients of t^j in $\delta^0(t)$ for $i - 1 < j \leq i$. In Section 3, we consider this case and deduce the following weighted version of Ehrhart Reciprocity (Theorem 3.7).

Theorem (Weighted Ehrhart Reciprocity). *Suppose the origin lies in the interior of P. For every rational number $-1 < k \leq 0$, $f_k^0(m)$ is either identically zero or a polynomial of degree d in $\mathbb{Q}[t]$ with positive leading coefficient. For any positive integer m,*

$$f_k^0(-m) = \begin{cases} (-1)^d f_{-1-k}^0(m) & \text{if } -1 < k < 0 \\ (-1)^d f_k^0(m-1) & \text{if } k = 0. \end{cases}$$

We show that Ehrhart Reciprocity for lattice polytopes is an immediate consequence (Corollary 3.8) as well as a result of Hibi (Corollary 3.11). In fact, Weighted Ehrhart Reciprocity implies Ehrhart Reciprocity for rational polytopes (Remark 3.10).

We now consider the geometric side of the story. Let $v_1, \ldots, v_r$ be the primitive integer vectors of Σ. For $i = 1, \ldots, r$, there is a positive integer a_i such that $a_i v_i$ lies in ∂P_0. The data $\boldsymbol{\Sigma} = (N, \Sigma, \{a_i v_i\})$ is called a *stacky fan*. We can associate to $\boldsymbol{\Sigma}$ a Deligne-Mumford stack $\mathcal{X} = \mathcal{X}(\boldsymbol{\Sigma})$ with coarse moduli space the toric variety $X = X(\Sigma)$ [5]. The theory of orbifold cohomology, developed by Chen and Ruan [7, 6], associates to $\mathcal{X}$ a finite-dimensional $\mathbb{Q}$-algebra $H_{\mathrm{orb}}^*(\mathcal{X}, \mathbb{Q})$, graded by $\mathbb{Q}$. In Section 4, we use a result of Borisov, Chen and Smith (Proposition 4.7 [5]) to

deduce our desired geometric interpretation of the coefficients of the δ-vector of P (Theorem 4.3).

Theorem. *The coefficient of t^j in $\delta^0(t)$ is equal to $\dim_{\mathbb{Q}} H^{2j}_{\text{orb}}(\mathcal{X}, \mathbb{Q})$. Moreover, the coefficient δ_i of t^i in the δ-vector $\delta_P(t)$ is a sum of dimensions of orbifold cohomology groups,*

$$\delta_i = \sum_{2i-2 < j \leq 2i} \dim_{\mathbb{Q}} H^j_{\text{orb}}(\mathcal{X}, \mathbb{Q}).$$

Using the above theorem, we show that Weighted Ehrhart Reciprocity is a consequence of Poincaré duality for orbifold cohomology (Remark 4.5).

Another corollary of the above theorem is that we may express the coefficients of the δ-vector of P as dimensions of orbifold cohomology groups of a $(d+1)$-dimensional orbifold (Theorem 4.6). More specifically, fix a lattice triangulation $\mathcal{T}$ of P and let σ be the cone over $P \times \{1\}$ in $(N \times \mathbb{Z})_{\mathbb{R}}$. If $\triangle$ denotes the simplicial fan refinement of σ induced by $\mathcal{T}$, then we may consider the corresponding toric variety $Y = Y(\triangle)$, with its canonical orbifold structure.

Theorem. *Let P be a d-dimensional lattice polytope and let $\mathcal{T}$ be a lattice triangulation of P corresponding to a $(d+1)$-dimensional toric variety Y as above. The Ehrhart δ-vector of P has the form*

$$\delta_P(t) = \sum_{i=0}^{d} \dim_{\mathbb{Q}} H^{2i}_{\text{orb}}(Y, \mathbb{Q}) t^i.$$

In the final section, we give a third proof of Weighted Ehrhart Reciprocity, generalising the toric proof of Ehrhart Reciprocity in [12]. More specifically, we show that Weighted Ehrhart Reciprocity is a consequence of Serre Duality as well as some vanishing theorems for ample divisors on toric varieties due to Mustaţă [22]. We note that this proof applies under the assumptions of the introduction, but not under the more general assumptions in Theorem 3.7.

We end the introduction with an example illustrating Weighted Ehrhart Reciprocity and showing how the weighted δ-vector gives rise to the Ehrhart δ-vector.

Example 1.1. Let $N = \mathbb{Z}^2$ and let P be the lattice polytope with vertices $(1,0), (0,2)$, $(-1,2)$, $(-2,1)$, $(-2,0)$ and $(0,-1)$. Since the origin lies in the interior of P, weighted Ehrhart Reciprocity holds and the weighted δ-vector $\delta^0(t)$ is symmetric. We can compute $\delta^0(t)$ using Lemma 3.13 and one can show that
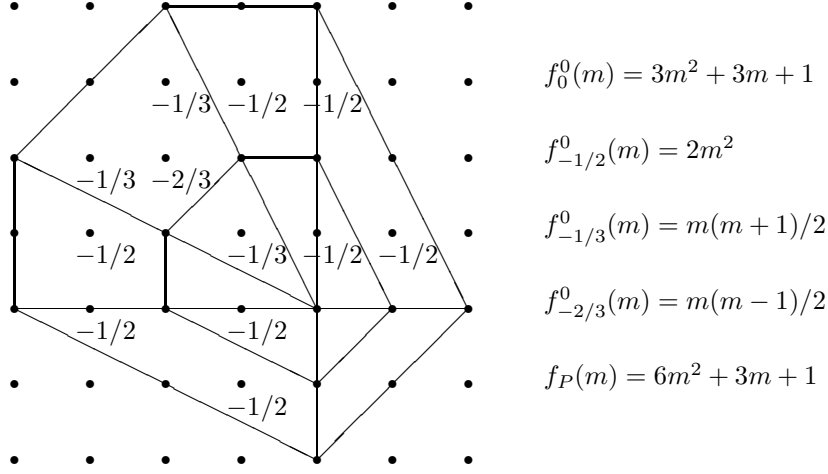
$$f_0^0(m) = |\partial P \cap N| m(m+1)/2 + 1$$

$$f_k^0(m) = (f_k^0(1) + f_{-1-k}^0(1))m^2/2 + (f_k^0(1) - f_{-1-k}^0(1))m/2 \text{ for } k \neq 0.$$

Note that the lattice points of weight 0 are precisely those lying on $\partial(mP)$ for some non-negative integer m. We have marked the lattice points of non-zero weight in $2P$.

$$\delta^0(t) = t^2 + 2t^{3/2} + t^{4/3} + 4t + t^{2/3} + 2t^{1/2} + 1$$

$$\delta_P(t) = 4t^2 + 7t + 1$$

$$f_0^0(m) = 3m^2 + 3m + 1$$

$$f_{-1/2}^0(m) = 2m^2$$

$$f_{-1/3}^0(m) = m(m+1)/2$$

$$f_{-2/3}^0(m) = m(m-1)/2$$

$$f_P(m) = 6m^2 + 3m + 1$$

We give a brief outline of the paper and note that we will use a more general setup than that in the introduction. In Section 2, we introduce the notion of a weighted δ-vector and describe some of its properties. We specialise to the case when $\lambda \equiv 0$ and prove weighted Ehrhart Reciprocity in Section 3. In Section 4, we consider orbifold cohomology and deduce our geometric interpretation of the coefficients of the δ-vector of P. In Section 5, we give a proof of Weighted Ehrhart Reciprocity using Serre Duality and vanishing theorems for ample divisors on toric varieties.

2. Weighted δ-Vectors

The goal of this section is to define weighted δ-vectors and study their properties. We will fix the following notation throughout the paper. Our setup will be slightly more general than that in the introduction (c.f. Remark 2.3). Let N be a lattice of rank d and set $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Let Σ be a simplicial, rational, d-dimensional fan in $N_\mathbb{R}$. We assume that the support $|\Sigma|$ of Σ in $N_\mathbb{R}$ is convex. Recall that Σ is *complete* if $|\Sigma| = N_\mathbb{R}$. Let $\rho_1, \ldots, \rho_r$ denote the rays of Σ, with primitive integer generators $v_1, \ldots, v_r$ in N. Fix elements $b_1, \ldots, b_r$ in N such that $b_i = a_i v_i$ for some positive integer a_i, for $i = 1, \ldots, r$. The data $\boldsymbol{\Sigma} = (N, \Sigma, \{b_i\})$ is called a *stacky fan* [5]. Let $\psi : |\Sigma| \to \mathbb{R}$ be the function that is $\mathbb{Q}$-linear on each cone of Σ and satisfies $\psi(b_i) = 1$ for $i = 1, \ldots, r$. We define

$$Q = Q_{\boldsymbol{\Sigma}} = \{v \in |\Sigma| \mid \psi(v) \leq 1\}.$$

Observe that Q need not be convex and that the union of the facets in the boundary ∂Q of Q not containing 0 is given by

$$\{v \in N_\mathbb{R} \mid \psi(v) = 1\}.$$

Hence ψ depends only on Q, not on the corresponding stacky fan.

Remark 2.1. A *pure lattice complex* of dimension d is a simplicial complex such that all maximal faces are lattice polytopes of dimension d [4]. For any cone τ in Σ, $Q \cap \tau$ is a lattice simplex containing the origin as a vertex. It follows that Q has the structure of a pure lattice complex. Conversely, let K be a pure lattice complex of dimension d in $N_{\mathbb{R}}$ such that the smallest cone containing K is convex. If every maximal simplex in K contains the origin as a vertex, then $K = Q_{\mathbf{\Sigma}}$ for some stacky fan $\mathbf{\Sigma}$ as above. More specifically, let Σ be the cone over the faces in K not containing the origin and make an appropriate choice of $\{b_i\}$.

For each positive integer m, let $f_Q(m)$ be the number of lattice points in mQ. Then $f_Q(m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of Q [4]. We write

$$(1) \qquad f_Q(m) = c_d m^d + c_{d-1} m^{d-1} + \cdots + c_0.$$

The generating series of the Ehrhart polynomial can be written in the form

$$\sum_{m \geq 0} f_Q(m) t^m = \delta_Q(t)/(1-t)^{d+1},$$

where $\delta_Q(t)$ is a polynomial of degree less than or equal to d with non-negative integer coefficients [27]. With a slight abuse of terminology, $\delta_Q(t)$ is called the *(Ehrhart) δ-vector* of Q. We write

$$(2) \qquad \delta_Q(t) = \delta_d t^d + \delta_{d-1} t^{d-1} + \cdots + \delta_0.$$

Remark 2.2. Consider a triple $(N, \triangle, \{b_i\})$ as above but without the assumption that $\triangle$ is simplicial. Suppose there exists a piecewise $\mathbb{Q}$-linear function $\psi : |\triangle| \to \mathbb{R}$ satisfying $\psi(b_i) = 1$, and set $Q = \{v \in |\triangle| \mid \psi(v) \leq 1\}$. There exists a simplicial fan Σ refining $\triangle$ and with the same rays as $\triangle$ (see, for example, [24]). If $\mathbf{\Sigma}$ denotes the stacky fan $(N, \Sigma, \{b_i\})$, then $Q = Q_{\mathbf{\Sigma}}$.

Remark 2.3. We consider the following important example. Let P be a lattice polytope and let α be a lattice point in P. After translating, we may assume that α is the origin. Let $\triangle$ be the fan over the faces of P not containing the origin. As in Remark 2.2, let Σ be a simplicial fan refining $\triangle$ and with the same rays as $\triangle$. With the appropriate choice of $\{b_i\}$, $P = Q_{\mathbf{\Sigma}}$. In the introduction, we state the main results of the paper in this context.

Let $\lambda : |\Sigma| \to \mathbb{R}$ be an arbitrary piecewise $\mathbb{Q}$-linear function with respect to Σ. We introduce the 'weight function' w_λ on $|\Sigma| \cap N$,

$$w_\lambda : |\Sigma| \cap N \to \mathbb{Q}$$

$$w_\lambda(v) = \psi(v) - \lceil \psi(v) \rceil + \lambda(v).$$

Note that when $\lambda \equiv 0$, the corresponding weight function w_0 is determined by Q. Think of w_λ as assigning a weight to every lattice point in $|\Sigma|$. For every rational number k and for every non-negative integer m, denote by $f_k^\lambda(m)$ the number of lattice points of weight k in mQ. Note that the Ehrhart polynomial of Q can be recovered as $f_Q(m) = \sum_{k \in \mathbb{Q}} f_k^\lambda(m)$. For every rational number k, consider the power series

$$\delta_k^\lambda(t) := (1-t)^{d+1} \sum_{m \geq 0} f_k^\lambda(m) t^m.$$

If $\lambda(v) \geq 0$ for all v in $|\Sigma|$, we will show that $\delta_k^\lambda(t)$ is a polynomial in t with integer coefficients (Corollary 2.9). The Ehrhart δ-vector of Q decomposes as $\delta_Q(t) = \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t)$. We define the *weighted δ-vector* of Q by

$$\delta^\lambda(s,t) := \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t)s^k.$$

It follows from the definition that the weighted δ-vector can be written as

$$\delta^\lambda(s,t) = (1-t)^{d+1} \sum_{m \geq 0} \Big(\sum_{v \in mQ \cap N} s^{w_\lambda(v)} \Big) t^m,$$

and hence $\delta^\lambda(s,t)$ is a well-defined element of $\mathbb{Z}[s^q \mid q \in \mathbb{Q}][[t]]$. Note that when $s = 1$, we recover the Ehrhart δ-vector $\delta^\lambda(1,t) = \delta_Q(t)$. Later, as in the introduction, we will consider the case when $\delta^\lambda(t,t)$ lies in $\mathbb{Z}[[t^{1/N}]]$ for some positive integer N. Our first aim is to express $\delta^\lambda(s,t)$ as a rational function in $\mathbb{Q}(s^{1/N}, t)$, for some positive integer N (Proposition 2.6).

For each cone τ in Σ, let Σ_τ be the simplicial fan in $(N/N_\tau)_{\mathbb{R}}$ with cones given by the projections of the cones in Σ containing τ. If τ is not contained in the boundary of $|\Sigma|$, then Σ_τ is complete. The h-vector of Σ_τ is given by

$$h_\tau(t) := \sum_{\tau \subseteq \sigma} t^{\dim \sigma - \dim \tau}(1-t)^{\operatorname{codim} \sigma}.$$

We will sometimes write $h_\Sigma(t)$ for $h_{\{0\}}(t)$. We will use the following standard lemma. For a combinatorial proof, we refer the reader to [28] and Lemma 1.3 [17]. We provide a geometric proof, deducing the result as a corollary of Lemma 4.1, which is proved independently.

Lemma 2.4. *For each cone τ in Σ, $h_\tau(t)$ is a polynomial of degree at most $\operatorname{codim} \tau$ with non-negative integer coefficients. Suppose that τ is not contained in the boundary of Σ. Then $h_\tau(t) = t^{\operatorname{codim} \tau} h_\tau(t^{-1})$ and the coefficients of $h_\tau(t)$ are positive integers.*

Proof. It follows from the definition that $h_\tau(t)$ is a polynomial of degree at most $\operatorname{codim} \tau$. Consider the simplicial toric variety $X = X(\Sigma_\tau)$. By Lemma 4.1, we can interpret the coefficient of t^i in $h_\tau(t)$ as the dimension of the $2i^{\text{th}}$ cohomology group of X. In particular, each coefficient is non-negative. If τ is not contained in the boundary of Σ, then X is complete. In this case, $h_\tau(t) = t^{\operatorname{codim} \tau} h_\tau(t^{-1})$ follows from Poincaré duality on X and the coefficients of $h_\tau(t)$ are positive by Lemma 4.1. $\qquad\square$

We define the *weighted h-vector $h_\tau^\lambda(s,t)$* of Σ_τ,

$$(3) \quad h_\tau^\lambda(s,t) := \sum_{\tau \subseteq \sigma} s^{\sum_{\rho_i \subseteq \sigma \setminus \tau} \lambda(b_i)} t^{\dim \sigma - \dim \tau}(1-t)^{\operatorname{codim} \sigma} \prod_{\rho_i \subseteq \sigma \setminus \tau} (1-t)/(1-s^{\lambda(b_i)}t).$$

The weighted h-vector is a rational function in $\mathbb{Q}(s^{1/N'}, t)$, for some positive integer N'. Note that if $\lambda \equiv 0$ or if we set $s = 1$, then $h_\tau^\lambda(s,t)$ is equal to the usual h-vector $h_\tau(t)$ of Σ_τ. By expanding and collecting terms, we have the following equality,

$$(4) \qquad t^{\operatorname{codim} \tau} h_\tau^\lambda(s^{-1}, t^{-1}) = \sum_{\tau \subseteq \sigma} (t-1)^{\operatorname{codim} \sigma} \prod_{\rho_i \subseteq \sigma \setminus \tau} (t-1)/(s^{\lambda(b_i)}t - 1).$$

Lemma 2.5. *Suppose that τ is a cone not contained in the boundary of Σ. Then*

$$h_\tau^\lambda(s,t) = t^{\operatorname{codim}\tau} h_\tau^\lambda(s^{-1}, t^{-1}).$$

Proof. This is an application of Möbius inversion (see, for example, [28]). Let $\mathcal{P}$ be the poset consisting of the cones in Σ_τ and of a maximal element $\{\Sigma_\tau\}$. By Lemma 2.4, $h_\tau(t) = t^{\operatorname{codim}\tau} h_\tau(t^{-1})$. Substituting $t = 0$ gives $\sum_{\tau \subseteq \sigma}(-1)^{\operatorname{codim}\sigma} = 1$. It follows that we can compute the Möbius function of $\mathcal{P}$ inductively. Möbius inversion says that if $f : \mathcal{P} \to A$ is a function, for some abelian group A, then

$$(5) \qquad f(\{\Sigma_\tau\}) = g(\{\Sigma_\tau\}) + \sum_{\tau \subseteq \sigma}(-1)^{\operatorname{codim}\sigma + 1} g(\sigma),$$

where, for each p in $\mathcal{P}$, $g(p) = \sum_{q \leq p} f(q)$.

We set $f(\{\Sigma_\tau\}) = 0$ and

$$f(\sigma) = (t-1)^{\operatorname{codim}\sigma} \prod_{\rho_i \subseteq \sigma \setminus \tau} (t-1)/(s^{\lambda(b_i)}t - 1).$$

Then $g(\{\Sigma_\tau\}) = t^{\operatorname{codim}\tau} h_\tau^\lambda(s^{-1}, t^{-1})$ by (4) and we calculate

$$g(\sigma) = \sum_{\tau \subseteq \sigma' \subseteq \sigma} f(\sigma')$$

$$= (t-1)^{\operatorname{codim}\tau} \sum_{\tau \subseteq \sigma' \subseteq \sigma} \prod_{\rho_i \subseteq \sigma' \setminus \tau} 1/(s^{\lambda(b_i)}t - 1)$$

$$= (t-1)^{\operatorname{codim}\tau} \prod_{\rho_i \subseteq \sigma \setminus \tau} (1 + 1/(s^{\lambda(b_i)}t - 1))$$

$$= s^{\sum_{\rho_i \in \sigma \setminus \tau} \lambda(b_i)} t^{\dim\sigma - \dim\tau} f(\sigma).$$

By (3) and (5),

$$h_\tau^\lambda(s,t) = \sum_{\tau \subseteq \sigma}(-1)^{\operatorname{codim}\sigma} g(\sigma) = g(\{\Sigma_\tau\}) = t^{\operatorname{codim}\tau} h_\tau^\lambda(s^{-1}, t^{-1}).$$

$\square$

As in [5], for each non-zero cone τ of Σ, set

$$(6) \qquad \operatorname{Box}(\boldsymbol{\tau}) = \{v \in N \mid v = \sum_{\rho_i \subseteq \tau} q_i b_i \text{ for some } 0 < q_i < 1\}.$$

We set $\operatorname{Box}(\{\mathbf{0}\}) = \{0\}$ and $\operatorname{Box}(\boldsymbol{\Sigma}) = \cup_{\tau \in \Sigma} \operatorname{Box}(\boldsymbol{\tau})$. Given any v in $|\Sigma| \cap N$, let $\sigma(v)$ be the cone in Σ containing v in its relative interior. As in [23, p.7], v has a unique decomposition

$$(7) \qquad v = \{v\} + v' + \sum_{\rho_i \subseteq \sigma(v) \setminus \tau} b_i,$$

where $\{v\}$ lies in $\operatorname{Box}(\boldsymbol{\tau})$ for some $\tau \subseteq \sigma(v)$ and v' is a linear combination of the $\{b_i \mid \rho_i \subseteq \sigma(v)\}$ with non-negative integer coefficients. We think of $\{v\}$ as the 'fractional part' of v. Using this decomposition, we compute a local formula for the weighted δ-vector of Q. The method of proof is the same as that of Theorem 1.3 in [23] and Theorem 1.2 in [25]. In fact, Proposition 2.6 can be deduced from Theorem 1.2 in [25].

Proposition 2.6. *The weighted δ-vector of Q is a rational function in $\mathbb{Q}(s^{1/N}, t)$, for some positive integer N, and has the form*

$$\delta^\lambda(s,t) = \sum_{\tau \in \Sigma} h_\tau^\lambda(s,t) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} s^{w_\lambda(v)} t^{\lceil \psi(v) \rceil} \prod_{\rho_i \subseteq \tau} (t-1)/(s^{\lambda(b_i)}t - 1).$$

Proof.

$$\delta^\lambda(s,t) = (1-t)^{d+1} \sum_{m \geq 0} \sum_{v \in mQ \cap N} s^{w_\lambda(v)} t^m$$

$$= (1-t)^{d+1} \sum_{v \in |\Sigma| \cap N} \sum_{m \geq \psi(v)} s^{w_\lambda(v)} t^m$$

$$= (1-t)^d \sum_{v \in |\Sigma| \cap N} s^{w_\lambda(v)} t^{\lceil \psi(v) \rceil}.$$

Now consider the decomposition (7). We have

$$w_\lambda(v) = w_\lambda(\{v\}) + w_\lambda(v') + \sum_{\rho_i \subseteq \sigma(v) \setminus \tau} \lambda(b_i).$$

Also

$$\lceil \psi(v) \rceil = \lceil \psi(\{v\}) \rceil + \psi(v') + \dim \sigma(v) - \dim \tau.$$

We obtain the following expression for $\delta^\lambda(t)$

$$(1-t)^d \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} s^{w_\lambda(v)} t^{\lceil \psi(v) \rceil} \sum_{\tau \subseteq \sigma} s^{\sum_{\rho_i \subseteq \sigma \setminus \tau} \lambda(b_i)} t^{\dim \sigma - \dim \tau} \prod_{\rho_i \subseteq \sigma} 1/(1 - s^{\lambda(b_i)}t).$$

Substituting in (3) and rearranging gives the result. $\square$

Example 2.7. If $\lambda \equiv 0$, then

$$\delta^0(s,t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} s^{\psi(v) - \lceil \psi(v) \rceil} t^{\lceil \psi(v) \rceil} h_\tau(t),$$

where $h_\tau(t)$ is the usual h-vector of Σ_τ. In this case, $\delta^0(s,t) \in \mathbb{Z}[s^{1/N}, s^{-1/N}, t]$, for some positive integer N. By Lemma 2.4, the coefficients of $h_\tau(t)$ are non-negative integers. Hence the coefficients of $\delta^0(s,t)$ are non-negative integers. Recall that, for any rational number $-1 < k \leq 0$, we can recover the polynomial $\delta_k^0(t)$ as the coefficient of s^k in $\delta^0(s,t)$. Note that the degree of $h_\tau(t)$ is at most $\mathrm{codim}\,\tau$ and $\lceil \psi(v) \rceil \leq \dim \tau$ for any v in $\mathrm{Box}(\boldsymbol{\tau})$. It follows that $\delta_k^0(t)$ is a polynomial in $\mathbb{Z}[t]$ of degree less than or equal to d, with non-negative coefficients.

Example 2.8. Recall that we can recover the Ehrhart δ-vector of Q as $\delta_Q(t) = \delta^\lambda(1,t)$. Then Proposition 2.6 gives the local formula of Betke and McMullen for $\delta_Q(t)$ (Theorem 1 in [4]),

$$\delta_Q(t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\lceil \psi(v) \rceil} h_\tau(t).$$

Note that the origin in N corresponds to a contribution of $h_0(t)$ in the above sum. It follows that $\delta_Q(t)$ is a polynomial of degree less than or equal to d with non-negative coefficients and constant term 1 [4]. If Σ is complete, we conclude from Lemma 2.4 that $\delta_Q(t)$ is a polynomial of degree d with positive integer coefficients [4].

Corollary 2.9. *Suppose that λ satisfies the additional condition*

$$(8) \qquad\qquad \lambda(v) \geq 0,$$

for every v in $|\Sigma|$. Then for every rational number k, $\delta_k^\lambda(t)$ is a polynomial in $\mathbb{Z}[t]$.

Proof. We know that $\delta_k^\lambda(t)$ is the coefficient of s^k in $\delta^\lambda(s,t)$ and is a power series in $\mathbb{Z}[[t]]$. We will show that it has bounded degree. By Proposition 2.6,

$$\delta^\lambda(s,t) = \sum_{\tau \in \Sigma} h_\tau^\lambda(s,t) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} s^{w_\lambda(v)} t^{\lceil \psi(v) \rceil} \prod_{\rho_i \subseteq \tau} (t-1)/(s^{\lambda(b_i)}t - 1).$$

Expanding the right hand side gives

$$(1-t)^d \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} s^{w_\lambda(v)} t^{\lceil \psi(v) \rceil} \sum_{\tau \subseteq \sigma} s^{\sum_{\rho_i \subseteq \sigma \setminus \tau} \lambda(b_i)} t^{\dim \sigma - \dim \tau} \prod_{\rho_i \subseteq \sigma} 1/(1 - s^{\lambda(b_i)}t).$$

If a monomial $t^l s^k$ appears in the expansion of this expression, then k must have the form

$$(9) \qquad\qquad k = w_\lambda(v) + \sum_{\rho_i \subseteq \sigma \setminus \tau} \lambda(b_i) + \sum_{i=1}^r \alpha_i \lambda(b_i),$$

for some v in $\mathrm{Box}(\boldsymbol{\Sigma})$ and α_i non-negative integers that are equal to zero if $\lambda(b_i) = 0$, and such that

$$l \leq \lceil \psi(v) \rceil + 2d + \sum_{i=1}^r \alpha_i.$$

It follows from Condition (8) that for a fixed k, there are only finitely many possibilities for α_i such that (9) holds. Therefore l is bounded. $\qquad\square$

By a standard argument, this is equivalent to the following corollary.

Corollary 2.10. *Suppose that λ satisfies the additional condition*

$$\lambda(v) \geq 0,$$

for every v in $|\Sigma|$. For every rational number k and for every m sufficiently large (depending on k), $f_k^\lambda(m)$ is a polynomial function in m, of degree less than or equal to d.

Proof. Fix a rational number k. By Corollary 2.9, we can write

$$F_k^\lambda(t) = P_k(t)/(1-t)^{d+1} + Q_k(t),$$

where $P_k(t) = p_{0,k} + p_{1,k}t + \cdots + p_{d,k}t^d$ and $Q_k(t)$ are polynomials. Expanding the right hand side gives

$$F_k^\lambda(t) = \sum_{j=0}^d p_{j,k} \sum_{m \geq 0} \binom{m+d}{d} t^{m+j} + Q_k(t).$$

Hence, for $m > \deg Q_k(t)$,

$$f_k^\lambda(m) = \sum_{j=0}^d p_{j,k} \binom{m+d-j}{d}.$$

$$\square$$

Example 2.11. As in Example 2.7, suppose that $\lambda \equiv 0$. We have seen that for every rational number k, $\delta_k^0(t)$ is a polynomial in $\mathbb{Z}[t]$ with non-negative coefficients, of degree less than or equal to d. We claim that $f_k^0(m)$ is either identically zero or a polynomial of degree d in $\mathbb{Q}[t]$ with positive leading coefficient. This follows from the above proof, which shows that $f_k^0(m)$ is a polynomial of degree less than or equal to d, and that the coefficient of t^d is $\sum_{j=0}^d p_{j,k}/d!$, where the $p_{j,k}$ are the non-negative coefficients of $\delta_k^0(t)$.

Suppose that λ satisfies the additional condition

(10) $$\lambda(b_i) > -1 \text{ for } i = 1, \ldots, r.$$

In this case we define

$$h_\tau^\lambda(t) := h_\tau^\lambda(t, t),$$
$$\delta^\lambda(t) := \delta^\lambda(t, t).$$

By Proposition 2.6, we have the following expression for $\delta^\lambda(t)$,

(11) $$\delta^\lambda(t) = \sum_{\tau \in \Sigma} h_\tau^\lambda(t) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} t^{\psi(v)+\lambda(v)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1).$$

It follows from (10) and (3) that there is a positive integer N' such that $h_\tau^\lambda(t)$ lies in $\mathbb{Z}[[t^{1/N'}]]$ for all τ in Σ. By (10), $w_\lambda(v) + \lceil \psi(v) \rceil = \psi(v) + \lambda(v)$ is non-negative for all v in $|\Sigma| \cap N$. Hence (11) implies that $\delta^\lambda(t) \in \mathbb{Z}[[t^{1/N}]]$, for some positive integer N. In this case, we will often abuse notation and call $\delta^\lambda(t)$ the *weighted* δ-vector associated to $\boldsymbol{\Sigma}$ and λ.

We now describe a well-known involution ι on $|\Sigma| \cap N$. Consider a cone τ in Σ and v in $\mathrm{Box}(\boldsymbol{\tau})$. Then v can be uniquely written in the form $v = \sum_{\rho_i \subseteq \tau} q_i b_i$, for some $0 < q_i < 1$. We define

(12) $$\iota = \iota_{\boldsymbol{\Sigma}} : \mathrm{Box}(\boldsymbol{\tau}) \to \mathrm{Box}(\boldsymbol{\tau})$$

$$\iota(v) = \sum_{\rho_i \subseteq \tau} (1 - q_i) b_i.$$

As in (7), every v in $|\Sigma| \cap N$ can be uniquely written in the form $v = \{v\} + \tilde{v}$, where $\{v\}$ is in $\mathrm{Box}(\boldsymbol{\tau})$ for some $\tau \subseteq \sigma(v)$ and $\tilde{v}$ is in $N_{\sigma(v)}$. Here $\sigma(v)$ is the cone of Σ containing v in its relative interior. Then ι extends to an involution on $|\Sigma| \cap N$

$$\iota = \iota_{\boldsymbol{\Sigma}} : |\Sigma| \cap N \to |\Sigma| \cap N$$

$$\iota(v) = \iota(\{v\}) + \tilde{v}.$$

Using (11) and after rearranging and collecting terms, we can write $t^d \delta^\lambda(t^{-1})$ as

$$\sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\mathrm{codim}\,\tau} h_\tau^\lambda(t^{-1}) t^{\sum_{\rho_i \subseteq \tau} \lambda(b_i)+\dim\tau-\psi(v)-\lambda(v)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1).$$

Note that for any v in $\mathrm{Box}(\boldsymbol{\tau})$,

$$\sum_{\rho_i \subseteq \tau} \lambda(b_i) + \dim\tau - \psi(v) - \lambda(v) = \psi(\iota(v)) + \lambda(\iota(v)),$$

where ι is the involution (12). Hence

(13) $$t^d \delta^\lambda(t^{-1}) = \sum_{\tau \in \Sigma} t^{\mathrm{codim}\,\tau} h_\tau^\lambda(t^{-1}) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} t^{\psi(v)+\lambda(v)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1).$$

Corollary 2.12. *Assume that Σ is a complete fan and $\lambda(b_i) > -1$ for $i = 1, \ldots, r$. Then*

$$\delta^\lambda(t) = t^d \delta^\lambda(t^{-1}).$$

Proof. By Lemma 2.5, for any cone τ in Σ, $h_\tau^\lambda(t) = t^{\operatorname{codim}\tau} h_\tau^\lambda(t^{-1})$. The result follows by comparing expressions (11) and (13). □

We have the following change of variables formula for weighted δ-vectors. A geometric proof involving motivic integration is given in [30]. We write $\psi = \psi_\Sigma$ and $\delta^\lambda(t) = \delta_\Sigma^\lambda(t)$.

Proposition 2.13. *Let N be a lattice of rank d. Let $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$ and $\triangle = (N, \triangle, \{b_j'\})$ be stacky fans such that $|\Sigma| = |\triangle|$. Let λ be a piecewise $\mathbb{Q}$-linear function with respect to Σ satisfying $\lambda(b_i) > -1$ for every b_i, and set*

$$\lambda' = \lambda + \psi_\Sigma - \psi_\triangle.$$

If λ' is piecewise $\mathbb{Q}$-linear with respect to $\triangle$ and satisfies $\lambda'(b_j') > -1$ for every b_j', then $\delta_\Sigma^\lambda(t) = \delta_\triangle^{\lambda'}(t)$.

Proof. By letting $s = t$ in the first calculation in the proof of Proposition 2.6, we see that

$$\delta_\Sigma^\lambda(t) = (1-t)^d \sum_{v \in |\Sigma| \cap N} t^{\psi_\Sigma(v) + \lambda(v)},$$

which gives,

$$\delta_\triangle^{\lambda'}(t) = (1-t)^d \sum_{v \in |\triangle| \cap N} t^{\psi_\triangle(v) + \lambda'(v)} = \delta_\Sigma^\lambda(t).$$

□

3. WEIGHTED EHRHART RECIPROCITY

The goal of this section is to investigate the case when $\lambda \equiv 0$. In this case, $\delta^0(t)$ is a polynomial of degree at d with rational powers and non-negative integer coefficients, and we can recover the Ehrhart δ-vector $\delta_Q(t)$ from $\delta^0(t)$. While $\delta_Q(t)$ is not symmetric in its coefficients, Corollary 2.12 implies if Σ is complete, then $\delta^0(t) = t^d \delta^0(t^{-1})$. The main point is that we can exploit this symmetry to deduce facts about the Ehrhart δ-vector.

Note that the weight function $w_0(v) = \psi(v) - \lceil \psi(v) \rceil$ takes values between -1 and 0. By Example 2.7, for each rational number $-1 < k \le 0$, we can write

$$\delta_k^0(t) = \delta_{d,k} t^d + \delta_{d-1,k} t^{d-1} + \cdots + \delta_{0,k},$$

for some non-negative integers $\delta_{i,k}$. Since the Ehrhart δ-vector decomposes as $\delta_Q(t) = \sum_{k \in (-1,0]} \delta_k^0(t)$, we have, with the notation of (2),

$$(14) \qquad \delta_i = \sum_{k \in (-1,0]} \delta_{i,k},$$

Throughout this section we will set $s = t$, so that the weighted δ-vector is given by

$$\delta^0(t) = \sum_{k \in (-1,0]} \delta_k^0(t) t^k = \sum_{k \in (-1,0]} \sum_{i=0}^d \delta_{i,k} t^{i+k}.$$

By (14), δ_i is the sum of the coefficients of t^j in $\delta^0(t)$ for $i-1 < j \leq i$. By Example 2.7,

$$(15) \qquad \delta^0(t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\psi(v)} h_\tau(t).$$

Later we will see that the coefficients of $\delta^0(t)$ are dimensions of orbifold Chow groups of a toric stack (Theorem 4.3).

Remark 3.1. The weight function w_0 and hence the weighted δ-vector $\delta^0(t)$ are determined by the underlying space of the lattice complex Q. Recall, from Remark 2.3, that any lattice polytope P, after translation, has the form $P = Q_{\boldsymbol{\Sigma}}$, for some stacky fan $\boldsymbol{\Sigma}$. In this case, $\delta^0(t)$ is determined by P and the choice of a lattice point α in P.

Remark 3.2. Note that the non-zero lattice points of weight 0 are those that lie in a facet of $\partial(mQ)$ not containing 0, for some positive integer m. Let ∂Q_0 denote the union of the facets in ∂Q not containing the origin. Consider the lattice $N \times \mathbb{Z}$ and the lattice complex

$$K_0 = \{(v, \mu) \in (N \times \mathbb{Z})_{\mathbb{R}} \mid 0 < \mu \leq 1,\, v \in \partial(\mu Q)_0\} \cup \{0\}.$$

We can interpret $f_0^0(m)$ as the number of lattice points in mK_0 and $\delta_0^0(t)$ as the δ-vector associated to K_0 (c.f. [29]).

Example 3.3. Let P be a lattice d-simplex containing the origin in its interior. Let Σ be the fan over the faces of the boundary of P, with the appropriate choice of $\{b_i\}$. One can verify that $h_\tau(t) = 1 + t + \cdots + t^{\mathrm{codim}\,\tau}$ for any cone τ in Σ. Hence

$$\delta^0(t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\psi(v)}(1 + t + \cdots + t^{\mathrm{codim}\,\tau}).$$

Corollary 3.4. *Suppose Σ is a complete fan. Then $\delta_0^0(t)$ has degree d and positive integer coefficients. For $-1 < k < 0$, we can write $\delta_k^0(t) = t\tilde{\delta}_k(t)$. Then*

$$\delta^0(t) = t^d \delta^0(t^{-1})$$

$$\delta_0^0(t) = t^d \delta_0^0(t^{-1})$$

$$\tilde{\delta}_k(t) = t^{d-1}\tilde{\delta}_{-1-k}(t^{-1}),$$

for $-1 < k < 0$.

Proof. By considering the contribution of $0 \in \mathrm{Box}\{0\}$ in (15) and using Lemma 2.4, we see that $\delta_0^0(t)$ has degree d and positive integer coefficients. By (15), $\delta_k^0(t)$ has no constant term for $-1 < k < 0$. By Corollary 2.12, we have $\delta^0(t) = t^d \delta^0(t^{-1})$. We can write

$$\delta^0(t) = \sum_{k \in (-1,0]} \delta_k^0(t) t^k.$$

$$t^d \delta^0(t^{-1}) = \sum_{k \in (-1,0]} t^d \delta_k^0(t^{-1}) t^{-k}$$

$$= t^d \delta_0^0(t^{-1}) + \sum_{k \in (-1,0)} t^{d+1} \delta_k^0(t^{-1}) t^{-1-k}$$

$$= t^d \delta_0^0(t^{-1}) + \sum_{k \in (-1,0)} t^{d+1} \delta_{-1-k}^0(t^{-1}) t^k.$$

Comparing the expressions above yields

$$\delta_0^0(t) = t^d \delta_0^0(t^{-1})$$

$$\delta_k^0(t) = t^{d+1} \delta_{-1-k}^0(t^{-1})$$

for $-1 < k < 0$. Since each $\delta_k^0(t)$ is a polynomial of degree less than or equal to d, the corollary follows. □

Remark 3.5. Suppose Σ is complete and set

$$a(t) = \delta_0^0(t), \; b(t) = \sum_{k \in (-1,0)} \tilde{\delta}_k(t).$$

Since $\delta_Q(t) = \sum_{-1 < k \leq 0} \delta_k^0(t)$, we have a decomposition

$$\delta_Q(t) = a(t) + tb(t).$$

This decomposition is due to Betke and McMullen (Theorem 5 [4]). We give an analogous result in the case when Σ is not necessarily complete in [29].

Remark 3.6. We can exploit the symmetry properties of the above decomposition to translate inequalities between the coefficients of $\delta_Q(t)$ into inequalities between the coefficients of $\delta_0^0(t)$. This is explained in [29] in a more general setting.

We give a reformulation of Corollary 3.4. We note that the $k = 0$ case can be deduced from Ehrhart Reciprocity (see Remark 3.9).

Theorem 3.7 (Weighted Ehrhart Reciprocity). *Suppose Σ is a complete fan. For every rational number $-1 < k \leq 0$, $f_k^0(m)$ is either identically zero or a polynomial of degree d in $\mathbb{Q}[t]$ with positive leading coefficient. For any positive integer m,*

$$f_k^0(-m) = \begin{cases} (-1)^d f_k^0(m-1) & \text{if } k = 0 \\ (-1)^d f_{-1-k}^0(m) & \text{if } -1 < k < 0. \end{cases}$$

Proof. Fix a rational number k. The first statement follows from Example 2.11. We see from the proof of Corollary 2.10 that

$$f_k^0(m) = \sum_{j=0}^{d} \delta_{j,k} \binom{m+d-j}{d}.$$

Hence

$$f_k^0(-m) = (-1)^d \sum_{j=0}^{d} \delta_{j,k} \binom{m+j-1}{d} \text{ for } m \geq 1.$$

We define

$$F_k(t) = \sum_{m \geq 0} f_k^0(m) t^m, \tilde{F}_k(t) = \sum_{m \geq 1} f_k^0(-m) t^m.$$

We compute

$$
\begin{aligned}
-F_k(t^{-1}) &= -\delta_k^0(t^{-1})/(1-t^{-1})^{d+1} \\
&= (-1)^d t^{d+1} \delta_k^0(t^{-1})/(1-t)^{d+1} \\
&= (-1)^d \sum_{j=0}^{d} \sum_{m \geq 0} \delta_{j,k} \binom{m+d}{d} t^{m+d+1-j} \\
&= \sum_{j=0}^{d} \sum_{m' \geq d+1-j} (-1)^d \delta_{j,k} \binom{m'+j-1}{d} t^{m'} \\
&= \sum_{m' \geq 1} \sum_{j=0}^{d} (-1)^d \delta_{j,k} \binom{m'+j-1}{d} t^{m'} \\
&= \tilde{F}_k(t).
\end{aligned}
$$

It follows from Corollary 3.4 (after substituting in definitions) that

$$
t F_0(t) = (-1)^{d+1} F_0(t^{-1})
$$
$$
F_k(t) = (-1)^{d+1} F_{-1-k}(t^{-1}) \text{ for } -1 < k < 0.
$$

The result follows by comparing coefficients of the expressions

$$
\tilde{F}_0(t) = (-1)^d t F_0(t)
$$
$$
\tilde{F}_k(t) = (-1)^d F_{-1-k}(t) \text{ for } -1 < k < 0.
$$

$\square$

The above result should be viewed as the weighted version of Ehrhart Reciprocity. In particular, we verify below that Ehrhart Reciprocity is a consequence. We emphasise that this proof can be interpreted as a reformulation of previous proofs of Ehrhart's theorem.

Corollary 3.8 (Ehrhart Reciprocity [11]). *For every positive integer m, $(-1)^d f_Q(-m)$ is equal to the number of lattice points in the interior of mQ.*

Proof. It follows from the definitions that

$$
(16) \qquad (-1)^d f_Q(-m) = \sum_{k \in (-1,0] \cap \mathbb{Q}} (-1)^d f_k^0(-m)
$$

$$
(17) \qquad |\operatorname{Int}(mQ) \cap N| = f_0^0(m-1) + \sum_{k \in (-1,0) \cap \mathbb{Q}} f_k^0(m).
$$

The fan Σ induces a lattice triangulation $\mathcal{T}$ of ∂Q. There exists a positive integer n and a lattice point α in the interior of nQ such that, after translating α to the origin, the collection of cones over the faces of $\mathcal{T}$ form a simplicial fan Σ'. It follows from (16), (17) and Example 2.7 that the functions $|\operatorname{Int}(mQ) \cap N|$ and $(-1)^d f_Q(-m)$ are polynomials in m. Hence, after replacing Q by a multiple and replacing Σ by Σ', we may assume that Σ is complete. The result now follows by applying Theorem 3.7 to (16) in order to obtain (17).

$\square$

Remark 3.9. This result was conjectured by Ehrhart in 1959 and proved by him in [11]. Another proof was given by MacDonald in [21]. If Σ is complete, then

(18) $$f_Q(m) - (-1)^d f_Q(-m) = f_0^0(m) - f_0^0(m-1),$$

since both sides are equal to the number of lattice points in $\partial(mQ)$. With the notation of (1), it follows that c_{d-1} is half the surface area of Q, normalised with respect to the sublattice on each facet of Q. Note that c_d is the normalised volume of Q in N. These facts were established in [10] and [20]. Using (18) and applying a short induction, Ehrhart Reciprocity implies that $f_0^0(-m) = (-1)^d f_0^0(m-1)$ for any positive integer m, which was proved in Theorem 3.7 (c.f. [29]).

Remark 3.10. In a similar way, one can show that Weighted Ehrhart Reciprocity implies Ehrhart Reciprocity for rational polytopes (see, for example, [28]). More specifically, fix a positive integer r and set $Q' = (1/r)Q$. For $0 \le l < r$, one verifies from the definitions that

$$f_{Q'}(l + mr) = \sum_{l/r-1 < k \le 0} f_k(m) + \sum_{-1 < k \le l/r-1} f_k(m+1).$$

It follows that $f_{Q'}(m)$ is a quasipolynomial with period dividing r [28]. As in the proof of Corollary 3.8, after replacing Q by sQ for some positive integer s coprime to r, we can apply Theorem 3.7 to deduce that, for any positive integer m, $(-1)^d f_{Q'}(-m)$ is equal to the number of lattice points in the interior of mQ'.

Let P be a rational polytope and fix a positive integer r such that rP is a lattice polytope. The function $f_P(m)$ is a quasipolynomial with period dividing r [28]. After replacing P by sP for some positive integer s coprime to r and translating by a lattice point, we may assume that P contains the origin. Setting $Q' = P$ and $Q = rP$, we deduce that $(-1)^d f_P(-m)$ is equal to the number of lattice points in the interior of mP.

The following result of Hibi was originally shown to be a consequence of Ehrhart Reciprocity [16]. In a similar way, it follows from Theorem 3.7. Recall that ψ is a piecewise $\mathbb{Q}$-linear function.

Corollary 3.11 ([16]). *If Σ is complete, then $\delta_Q(t) = t^d \delta_Q(t^{-1})$ if and only if ψ is a piecewise linear function.*

Proof. Observe that ψ is piecewise linear if and only if $w_0 \equiv 0$. If $w_0 \equiv 0$, then by Corollary 3.4, $\delta_Q(t) = \delta^0(t) = t^d \delta^0(t^{-1}) = t^d \delta_Q(t^{-1})$. Conversely, suppose that $\delta_Q(t) = t^d \delta_Q(t^{-1})$. Assume w_0 is not identically 0. Choose $j+1$ minimal such that $\delta_{j+1,k} > 0$ for some $-1 < k < 0$. By Corollary 3.4 and (14),

$$\delta_j = \delta_{j,0} + \sum_{k \in (-1,0)} \delta_{j,k} = \delta_{j,0}.$$

$$\delta_{d-j} = \delta_{d-j,0} + \sum_{k \in (-1,0)} \delta_{d-j,k} = \delta_{j,0} + \sum_{k \in (-1,0)} \delta_{j+1,k} > \delta_j.$$

This is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.12. If P is a lattice polytope then δ_0 is 1 and δ_d is the number of interior lattice points of P. A lattice polytope P is *reflexive* if it contains the origin in its interior and ψ is piecewise linear. Then Corollary 3.11 says that $\delta_P(t) = t^d \delta_P(t^{-1})$ if and only if P is the translate of a reflexive polytope [16].

We conclude this section by proving a lemma which allows us to compute examples when $d = 2$ and Σ is complete, and then presenting a corresponding example. Recall that ∂Q_0 denotes the union of the facets of ∂Q not containing the origin.

Lemma 3.13. *The weighted δ-vector $\delta^0(t)$ is a polynomial of degree less than or equal to d with rational powers and non-negative integer coefficients. If Σ is complete, then $\delta^0(t) = t^d \delta^0(t^{-1})$.*

  (1) $\delta^0(1) = d! \operatorname{vol}_d(Q)$.
  (2) *The constant coefficient in $\delta^0(t)$ is 1.*
  (3) *For $0 < l < 1$, the coefficient of t^l in $\delta^0(t)$ is $|\{v \in Q \cap N \mid \psi(v) = l\}|$.*
  (4) *The coefficient of t in $\delta^0(t)$ is $|\partial Q_0 \cap N| - d$.*

Proof. We established the initial claims in Corollary 2.10 and Corollary 3.4. We showed in the proof of Corollary 2.10 that $f_k^0(m)$ is a polynomial of degree d with leading coefficient $\sum_{j=0}^{d} \delta_{j,k}/d!$. Hence

$$d! \operatorname{vol}_d(Q) = \sum_{k \in (-1,0]} \sum_{j=0}^{d} \delta_{j,k} = \delta^0(1).$$

For the other claims we compare both sides of the expression

$$\delta^0(t) = (1-t)^{d+1} \sum_{m \geq 0} \sum_{v \in mQ \cap N} t^{w_0(v)+m}.$$

The constant coefficient on the right hand side is 1. For $0 < l < 1$, the coefficient on the right hand side is $|\{v \in Q \cap N \mid w_0(v) + 1 = l\}|$. Note that if v lies in $Q \cap N$ and $w_0(v) \neq 0$ then $\lceil \psi(v) \rceil = 1$. Finally, the coefficient of t on the right hand side is $|\{v \in Q \cap N \mid w_0(v) = 0\}| - (d+1)$. The only elements of $Q \cap N$ of weight zero are the origin and the elements of $\partial Q_0 \cap N$. $\qquad\square$

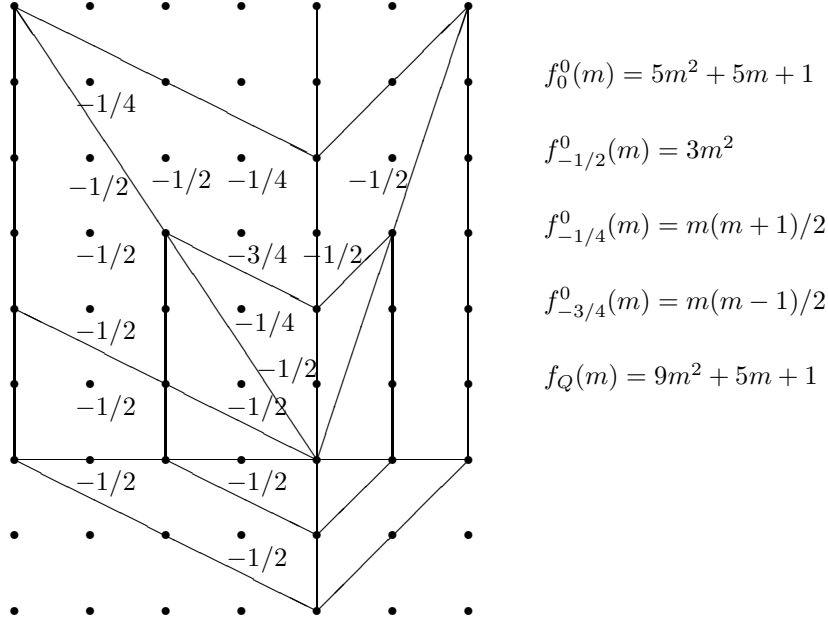*Remark* 3.14. When $d = 2$ and Σ is complete, one can show that

$$f_0^0(m) = |\partial Q \cap N| m(m+1)/2 + 1$$

$$f_k^0(m) = (f_k^0(1) + f_{-1-k}^0(1)) m^2/2 + (f_k^0(1) - f_{-1-k}^0(1)) m/2 \text{ for } k \neq 0.$$

Example 3.15. Let $N = \mathbb{Z}^2$ and let Σ be the complete fan with primitive integer vectors $(1,0),(1,3), (0,1), (-2,3),(-2,1),(-1,0)$ and $(0,-1)$, and set a_i to be $1, 1, 2, 1, 1, 2$ and 1 respectively. Since Σ is a complete fan, weighted Ehrhart Reciprocity holds and the weighted δ-vector $\delta^0(t)$ is symmetric. The example is illustrated in the diagram below, in which we have marked the lattice points of non-zero weight in $2Q$. The computations were made using Lemma 3.13 and Remark 3.14. Observe that removing the ray through $(-2,1)$ does not affect the results below.

$$\delta^0(t) = t^2 + 3t^{3/2} + t^{5/4} + 8t + t^{3/4} + 3t^{1/2} + 1$$

$$\delta_Q(t) = 5t^2 + 12t + 1$$

$$f_0^0(m) = 5m^2 + 5m + 1$$

$$f_{-1/2}^0(m) = 3m^2$$

$$f_{-1/4}^0(m) = m(m+1)/2$$

$$f_{-3/4}^0(m) = m(m-1)/2$$

$$f_Q(m) = 9m^2 + 5m + 1$$

4. Orbifold Cohomology

We will now prove our geometric interpretation of the coefficients of the Ehrhart δ-vector. Recall that Σ is a simplicial, d-dimensional fan with convex support. In this case, we have a combinatorial description of the Betti numbers of the corresponding toric variety $X = X(\Sigma)$.

Lemma 4.1. *Let Σ is a simplicial, d-dimensional fan with convex support. Let $X = X(\Sigma)$ be the toric variety associated to Σ. Then X has no odd cohomology over $\mathbb{Q}$ and $\dim H^{2i}(X, \mathbb{Q})$ is equal to the coefficient of t^i in the h-vector $h_\Sigma(t)$ of Σ. If Σ is complete, then $\dim H^{2i}(X, \mathbb{Q}) > 0$ for $i = 0, \ldots, d$.*

Proof. The case when Σ is complete is due to Danilov (Theorem 10.8 [8]). Suppose Σ is not complete. Let ρ be a ray in the interior of $-|\Sigma|$. Let $\triangle$ be the fan with cones given by the cones of Σ as well as the cones generated by ρ and a face of Σ contained in $\partial|\Sigma|$. Consider the complete, simplicial toric variety $Y = Y(\triangle)$. Let $D = D(\triangle_\rho)$ be the $\mathbb{Q}$-Cartier torus-invariant divisor corresponding to ρ. Then D is simplicial and complete and $Y \smallsetminus D = X$. By considering the long exact sequence of cohomology with compact supports, we have a diagram,

$$A^i(Y, \mathbb{Q}) \xrightarrow{\alpha} A^i(D, \mathbb{Q}) \longrightarrow 0$$

$$0 \longrightarrow H_c^{2i}(X, \mathbb{Q}) \longrightarrow H_c^{2i}(Y, \mathbb{Q}) \longrightarrow H_c^{2i}(D, \mathbb{Q}) \longrightarrow H_c^{2i+1}(X, \mathbb{Q}) \longrightarrow 0.$$

Since a complete toric variety has no odd cohomology, the bottom row is exact. The vertical maps take a cycle to its corresponding cohomology class and both maps are isomorphisms by Theorem 10.8 of [8]. The map α 'restricts' cycles of Y to cycles on the $\mathbb{Q}$-Cartier divisor D (p33 [13]). Let γ be a cone in $\partial|\Sigma|$ corresponding to a T-invariant subvariety $V(\gamma)$ of Y. If we set $\sigma = \gamma + \rho$, then σ corresponds

to a T-invariant subvariety $V_\rho(\sigma)$ of D and the set-theoretic intersection of $V(\gamma)$ and D is $V_\rho(\sigma)$. It follows that $\alpha([V(\gamma)])$ is a positive multiple of $[V_\rho(\sigma)]$ [12]. By Proposition 10.3 of [8], $A^*(D, \mathbb{Q})$ is generated by classes of the form $[V_\rho(\sigma)]$ and hence α is surjective and the diagram above has exact rows. We conclude that $H_c^{2i+1}(X, \mathbb{Q}) = 0$ and $\dim H_c^{2i}(X, \mathbb{Q})$ is equal to the ith coefficient of

$$t^d h_\triangle(t^{-1}) - t^d h_{\triangle_\rho}(t^{-1}) = \sum_{\tau \in \Sigma}(t-1)^{\operatorname{codim}\tau} = t^d h_\Sigma(t^{-1}).$$

By Poincaré duality, $\dim H^{2i}(X, \mathbb{Q})$ is equal to the coefficient of t^i in $h_\Sigma(t)$. $\square$

We can associate to the stacky fan $\boldsymbol{\Sigma} = (N, \Sigma, \{b_i\})$ a Deligne-Mumford toric stack $\mathcal{X} = \mathcal{X}(\boldsymbol{\Sigma})$ over $\mathbb{C}$ with coarse moduli space $X = X(\Sigma)$ [5]. Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_t$ denote the connected components of the corresponding inertia stack $\mathcal{IX}$. There is a degree shifting function

$$\iota_{\mathcal{X}} : \mathcal{IX} \to \mathbb{Q},$$

which is constant on connected components. Let $\overline{\mathcal{Y}}_i$ denote the coarse moduli space of $\mathcal{Y}_i$. For $i \in \mathbb{Q}$, Chen and Ruan [7] defined the ith orbifold cohomology group of $\mathcal{X}$ by

$$H_{\operatorname{orb}}^i(\mathcal{X}, \mathbb{Q}) = \bigoplus_{j=1}^t H^{i - 2\iota_{\mathcal{X}}(\mathcal{Y}_j)}(\overline{\mathcal{Y}}_j, \mathbb{Q}).$$

Similarly, we can consider the orbifold cohomology $H_{\operatorname{orb,c}}^*(\mathcal{X}, \mathbb{Q})$ of $\mathcal{X}$ with compact support. Chen and Ruan established Poincaré duality between $H_{\operatorname{orb}}^*(\mathcal{X}, \mathbb{Q})$ and $H_{\operatorname{orb,c}}^*(\mathcal{X}, \mathbb{Q})$ (Proposition 3.3.1 [7]).

Borisov, Chen and Smith (Proposition 4.7 [5]) show that the connected components of $\mathcal{IX}$ are indexed by the elements of $\operatorname{Box}(\boldsymbol{\Sigma})$. Moreover, if v in $\operatorname{Box}(\boldsymbol{\tau})$ corresponds to the connected component $\mathcal{Y}_v$, then $\iota_{\mathcal{X}}(\mathcal{Y}_v) = \psi(v)$ and $\overline{\mathcal{Y}}_v = X(\Sigma_\tau)$. Hence

$$(19) \qquad H_{\operatorname{orb}}^{2i}(\mathcal{X}, \mathbb{Q}) = \bigoplus_{\tau \in \Sigma} \bigoplus_{v \in \operatorname{Box}(\boldsymbol{\tau})} H^{2(i - \psi(v))}(X(\Sigma_\tau), \mathbb{Q}).$$

Remark 4.2. Similarly, Abramovich, Graber and Vistoli [1] defined the *orbifold Chow ring* $A_{\operatorname{orb}}^*(\mathcal{X}, \mathbb{Q})$ of $\mathcal{X}$. For $i \in \mathbb{Q}$,

$$A_{\operatorname{orb}}^i(\mathcal{X}, \mathbb{Q}) = \bigoplus_{j=1}^t H^{i - \iota_{\mathcal{X}}(\mathcal{Y}_j)}(\overline{\mathcal{Y}}_j, \mathbb{Q}).$$

The cohomology ring with rational coefficients and Chow ring with rational coefficients of a simplicial, complete toric variety are isomorphic (Theorem 10.8 [8]). Hence, if Σ is complete, then $A_{\operatorname{orb}}^i(\mathcal{X}, \mathbb{Q}) \cong H_{\operatorname{orb,c}}^{2i}(\mathcal{X}, \mathbb{Q})$.

We finally arrive at the main result of this section. The case when $w_0 \equiv 0$ is proved in [23].

Theorem 4.3. *The coefficient of t^j in $\delta^0(t)$ is equal to $\dim_{\mathbb{Q}} H_{\operatorname{orb}}^{2j}(\mathcal{X}(\boldsymbol{\Sigma}), \mathbb{Q})$. Moreover, the coefficient δ_i of t^i in the δ-vector $\delta_Q(t)$ is a sum of dimensions of orbifold cohomology groups,*

$$\delta_i = \sum_{2i - 2 < j \le 2i} \dim_{\mathbb{Q}} H_{\operatorname{orb}}^j(\mathcal{X}(\boldsymbol{\Sigma}), \mathbb{Q}).$$

Proof. By (15),

$$\delta^0(t) = \sum_{\tau \in \Sigma} \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} h_\tau(t) t^{\psi(v)}.$$

The first assertion now follows from (19) and Lemma 4.1. The second statement then follows from (14). □

Remark 4.4. By Poincaré duality, the coefficient of t^j in $t^d \delta^0(t^{-1})$ is equal to the dimension of the $(2j)^{\mathrm{th}}$ orbifold cohomology group of $\mathcal{X}$ with compact support.

Remark 4.5. When Σ is complete, we showed in Corollary 3.4 that

$$\delta^0(t) = t^d \delta^0(t^{-1}).$$

By Theorem 4.3, we can interpret this symmetry as a consequence of Poincaré duality for orbifold cohomology. In particular, since Weighted Ehrhart Reciprocity (Theorem 3.7) is equivalent to Corollary 3.4, this provides a geometric proof of Weighted Ehrhart Reciprocity.

A corollary of Theorem 4.3 is the following result which interprets the coefficients of the Ehrhart δ-vector of a lattice polytope as dimensions of orbifold cohomology groups of a $(d+1)$-dimensional orbifold. More specifically, let P be a d-dimensional lattice polytope in N and fix a lattice triangulation $\mathcal{T}$ of P. If σ denotes the cone over $P \times \{1\}$ in $(N \times \mathbb{Z})_{\mathbb{R}}$, then $\mathcal{T}$ determines a simplicial fan refinement $\triangle$ of σ. The corresponding toric variety $Y = Y(\triangle)$ has a canonical stack structure: if $w_1, \ldots, w_s$ are the primitive integer vectors of the rays of $\triangle$, then the corresponding stacky fan is $(N \times \mathbb{Z}, \triangle, \{w_i\})$. We will write $H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q})$ for the $2i^{\mathrm{th}}$ orbifold cohomology group of the canonical stack associate to Y.

Theorem 4.6. *Let P be a d-dimensional lattice polytope and let $\mathcal{T}$ be a lattice triangulation of P corresponding to a $(d+1)$-dimensional toric variety Y as above. The Ehrhart δ-vector of P has the form*

$$\delta_P(t) = \sum_{i=0}^d \dim_{\mathbb{Q}} H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q}) t^i.$$

Proof. With the notation of the previous discussion, let $\mathcal{Y}$ denote the toric stack associated to the stacky fan $(N \times \mathbb{Z}, \triangle, \{w_i\})$. In this case, $\psi : |\triangle| \to \mathbb{R}$ is the restriction of the projection $N_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ to $|\triangle|$, and hence $Q = \{v \in |\triangle| \mid \psi(v) \le 1\}$ is the convex hull of $P \times \{1\}$ and the origin, called the *pyramid* over P. Since the weight function $w_0(v) = \psi(v) - \lceil \psi(v) \rceil$ is identically zero on $|\triangle| \cap (N \times \mathbb{Z})$, the weighted δ-vector $\delta^0(t)$ is just the usual δ-vector $\delta_Q(t)$. It is a standard fact that $\delta_P(t) = \delta_Q(t)$ (see, for example, Remark 2.6 [2]). On the other hand, Theorem 4.3 implies that $\delta_P(t) = \delta_Q(t) = \delta^0(t) = \sum_{i=0}^d \dim_{\mathbb{Q}} H^{2i}_{\mathrm{orb}}(\mathcal{Y}, \mathbb{Q}) t^i$. □

Remark 4.7. If $\mathcal{T}$ is a unimodular triangulation of P, then Y is smooth and $H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q}) = H^{2i}(Y, \mathbb{Q})$. In this case, the above theorem and Lemma 4.1 imply the well-known fact that $\delta_P(t)$ is equal to the h-vector of $\mathcal{T}$ [15].

5. A TORIC PROOF OF WEIGHTED EHRHART RECIPROCITY

We have provided a combinatorial proof of Weighted Ehrhart Reciprocity (Theorem 3.7) as well as a geometric proof via orbifold cohomology (Remark 4.5). In this section, we give a third proof in the case when Q is a lattice polytope. More

specifically, we show that Weighted Ehrhart Reciprocity can be deduced from Serre
Duality as well as some vanishing theorems for ample divisors on toric varieties due
to Mustaţă [22]. This proof generalises the toric proof of Ehrhart Reciprocity given
in Section 4.4 of [12].

Throughout this section we will assume that Σ is a complete fan and

$$P := Q = \{v \in N_{\mathbb{R}} \mid \psi(v) \leq 1\}$$

is a lattice polytope (containing the origin in its interior). By definition, for every
rational number $-1 < k \leq 0$ and for any positive integer m,

(20) $$f_k^0(m) - f_k^0(m-1) = |\partial(m+k)P \cap N|.$$

Also, $f_0^0(0) = 1$ and $f_k^0(0) = 0$ for $-1 < k < 0$. Our goal is to provide a toric proof
of the following version of Weighted Ehrhart Reciprocity.

Theorem 5.1. *Let P be a d-dimensional lattice polytope containing the origin in
its interior. For every rational number $-1 < k \leq 0$, $f_k^0(m)$ is a polynomial in m of
degree at most d and for any positive integer m,*

$$f_k^0(-m) = \begin{cases} (-1)^d f_{-1-k}^0(m) & \text{if } -1 < k < 0 \\ (-1)^d f_k^0(m-1) & \text{if } k = 0. \end{cases}$$

We first recall some facts about toric varieties and refer the reader to [12] for
the relevant details. A d-dimensional lattice polytope P in N, containing the origin
in its interior, determines a d-dimensional, projective toric variety Y over $\mathbb{C}$ and
an effective ample torus-invariant divisor D on Y. If $M = \text{Hom}(N, \mathbb{Z})$ denotes the
dual lattice to N, then the normal fan to P in $M_{\mathbb{R}}$ determines the toric variety Y.
Let $u_1, \ldots, u_s$ denote the primitive integer vectors along the rays of the normal fan,
corresponding to the torus-invariant prime divisors $D_1, \ldots, D_s$ of Y. If we write
$D = \sum_{i=1}^s a_i D_i$, then $a_i = -\min_{v \in P \cap N} \langle u_i, v \rangle \in \mathbb{Z}_{>0}$. Given a torus-invariant
$\mathbb{Q}$-divisor $E = \sum_{i=1}^s b_i D_i$, consider the (possibly empty) polytope

$$P_E := \{v \in N_{\mathbb{R}} \mid \langle u_i, v \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, s\}.$$

It can be verified that $\lambda P = P_{\lambda D}$ for any rational number $\lambda > 0$. Every lattice point
v in N corresponds to a character χ^v on the torus contained in Y. In particular,
we may view χ^v as a rational function on Y. If we let $\lfloor E \rfloor = \sum_{i=1}^s \lfloor b_i \rfloor D_i$ denote
the round down of E, then the global sections of E are given by

(21) $$H^0(Y, \mathcal{O}(\lfloor E \rfloor)) = \bigoplus_{v \in P_E \cap N} \mathbb{C}\chi^v.$$

In particular, $\dim H^0(Y, \mathcal{O}(\lfloor E \rfloor)) = |P_E \cap N|$. We identify the canonical divisor of
Y with $K_Y = -\sum_{i=1}^s D_i$ and write $\lceil E \rceil = \sum_{i=1}^s \lceil b_i \rceil D_i$ for the round up of E.

Lemma 5.2. *If $E = \sum_{i=1}^s b_i D_i$ is a torus-invariant $\mathbb{Q}$-divisor on Y then*

$$H^0(Y, \mathcal{O}(K_Y + \lceil E \rceil)) = \bigoplus_{v \in \text{Int } P_E \cap N} \mathbb{C}\chi^v.$$

In particular, for each $-1 < k \leq 0$ and for every positive integer m,

$$f_k^0(m) - f_k^0(m-1) = \dim H^0(Y, \mathcal{O}(\lfloor(m+k)D\rfloor)) - \dim H^0(Y, \mathcal{O}(K_Y + \lceil(m+k)D\rceil)).$$

Proof. Observe that v in N lies in the interior of P_E if and only if

$$\langle u_i, v \rangle + b_i > 0 \text{ for } i = 1, \ldots, s.$$

This condition holds if and only if

$$\langle u_i, v \rangle + \lceil b_i \rceil - 1 \geq 0 \text{ for } i = 1, \ldots, s.$$

We conclude that $\operatorname{Int} P_E \cap N = P_{K_Y + \lceil E \rceil} \cap N$ and the first statement follows from (21). By (20), for each $-1 < k \leq 0$ and for every positive integer m,

$$f_k^0(m) - f_k^0(m-1) = |(m+k)P \cap N| - |\operatorname{Int}(m+k)P \cap N|$$
$$= \dim H^0(Y, \mathcal{O}(\lfloor (m+k)D \rfloor)) - \dim H^0(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil)).$$

$\square$

We recall the following vanishing theorem due to Mustaţă. A $\mathbb{Q}$-divisor E on Y is ample if mE is an ample divisor for some positive integer m.

Theorem 5.3 (Corollary 2.5 [22]). *Let Y be a projective toric variety and let E be an ample torus-invariant $\mathbb{Q}$-divisor on Y. For $i > 0$,*

(1) $H^i(Y, \mathcal{O}(K_Y + \lceil E \rceil)) = 0$ *(Kawamata-Viehweg vanishing)*
(2) $H^i(Y, \mathcal{O}(\lfloor E \rfloor)) = 0$.

For any divisor D' on Y, let $\chi(Y, D') = \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{O}(D'))$ denote the Euler characteristic of D'. By the above vanishing theorem and Lemma 5.2, for any positive integer m,

$$(22) \quad f_k^0(m) - f_k^0(m-1) = \chi(Y, \mathcal{O}(\lfloor (m+k)D \rfloor)) - \chi(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil)).$$

The following fact is due to Snapper.

Theorem 5.4 ([26]). *Let X be a complete variety of dimension d over an algebraically closed field k. If $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{L}$ is a line bundle on X, then there is a polynomial $Q(t)$ of degree at most d such that $Q(m) = \chi(X, \mathcal{F} \otimes \mathcal{L}^m)$ for every integer m.*

If we apply the above result to the coherent sheaf $\mathcal{O}(\lfloor kD \rfloor)$ and the line bundle $\mathcal{O}(D)$ on Y, we deduce that there is a polynomial $Q_1(t)$ of degree at most d such that $Q_1(m) = \chi(Y, \mathcal{O}(\lfloor (m+k)D \rfloor))$ for each integer m. Moreover, the coefficient of t^d in $Q_1(t)$ is the intersection number $\operatorname{rk}(\mathcal{O}(\lfloor kD \rfloor)) \cdot D^d/d! = D^d/d! > 0$ [19]. Similarly, there is a polynomial $Q_2(t)$ of degree d with leading term $D^d/d!$ such that $Q_2(m) = \chi(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil))$ for each integer m. By (22), for any positive integer m, $f_k^0(m) - f_k^0(m-1) = Q_1(m) - Q_2(m)$. It now follows from standard arguments (see, for example, p. 49 [14]) that $f_k^0(m)$ is a polynomial in m of degree at most d.

We have the following application of Serre Duality.

Lemma 5.5. *If $S_k(m) = f_k^0(m) - f_k^0(m-1)$ then*

$$(-1)^{d+1} S_k(-m) = \begin{cases} S_{-1-k}(m+1) & \text{if } -1 < k < 0 \\ S_k(m) & \text{if } k = 0. \end{cases}$$

Proof. Since both sides of (22) are polynomials in m, $(-1)^{d+1} S_k(-m)$ is equal to

$$-(-1)^d \chi(Y, \mathcal{O}(\lfloor (-m+k)D \rfloor)) + (-1)^d \chi(Y, \mathcal{O}(K_Y + \lceil (-m+k)D \rceil)).$$

By Serre Duality (see, for example, Corollary 3.7.7 [14]), this is equal to

$$-\chi(Y, \mathcal{O}(K_Y - \lfloor (-m+k)D \rfloor)) + \chi(Y, \mathcal{O}(-\lceil (-m+k)D \rceil)).$$

Since for any real number a, $-\lfloor -a \rfloor = \lceil a \rceil$,

$$(-1)^{d+1}S_k(-m) = -\chi(Y, \mathcal{O}(K_Y + \lceil (m-k)D \rceil)) + \chi(Y, \mathcal{O}(\lfloor (m-k)D \rfloor)).$$

When $k = 0$, we get $(-1)^{d+1}S_0(-m) = S_0(m)$. For $-1 < k \leq 0$, after writing $m - k = m + 1 + (-1 - k)$, we see that $(-1)^{d+1}S_k(-m) = S_{-1-k}(m+1)$. $\qquad \square$

We will now complete our proof of Theorem 5.1 by induction on m.

Proof. We have seen that for each $-1 < k \leq 0$, $f_k^0(m)$ is a polynomial in m of degree at most d. We will first prove Theorem 5.1 in the case when $m = 1$. Recall that $f_0^0(0) = 1$ and $f_k^0(0) = 0$ for $-1 < k < 0$. By (22) and Serre Duality,

$$f_0^0(0) - f_0^0(-1) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}(K_Y)) = (1 - (-1)^d)\chi(Y, \mathcal{O}_Y).$$

It follows from Theorem 5.3 that $\chi(Y, \mathcal{O}_Y) = 1$ and we conclude that $f_0^0(-1) = (-1)^d = (-1)^d f_0^0(0)$ as desired. When $-1 < k < 0$, Lemma 5.5 implies that $(-1)^{d+1}S_k(0) = S_{-1-k}(1)$. That is,

$$(-1)^{d+1}(f_k^0(0) - f_k^0(-1)) = f_{-1-k}^0(1) - f_{-1-k}^0(0),$$

and hence $(-1)^d f_k^0(-1) = f_{-1-k}^0(1)$. This completes the proof when $m = 1$.

Now consider the case when $m > 1$. By Lemma 5.5 and induction on m,

$$\begin{aligned} (-1)^d f_0^0(-m) &= (-1)^{d+1}S_0(-m+1) + (-1)^d f_0^0(-(m-1)) \\ &= S_0(m-1) + f_0^0(m-2) \\ &= f_0^0(m-1). \end{aligned}$$

Similarly, when $-1 < k < 0$,

$$\begin{aligned} (-1)^d f_k^0(-m) &= (-1)^{d+1}S_k(-m+1) + (-1)^d f_k^0(-(m-1)) \\ &= S_{-1-k}(m) + f_{-1-k}^0(m-1) \\ &= f_{-1-k}^0(m). \end{aligned}$$

$$\square$$

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